

Recitation 1 Solutions

Problem 1.1

Tom rolls two six-sided dice.

- (a) Determine the fraction of possible outcomes in which the first roll is odd.

Solution: There are 6 possible outcomes for the first roll, and 6 possible outcomes for the second roll, producing 36 possible two-roll outcomes. There are 3 outcomes of the first roll in which the result is an odd number ('1', '3' and '5'), representing $1/2$ of all outcomes for the first roll. Since each outcome of the first roll is coupled with exactly 6 outcomes of the second roll, we conclude that the fraction of two-roll outcomes in which the first roll is odd is also $1/2$.

Interestingly, the answer does not depend on the number of faces for the second die. We will later see that this is related to what we will call independence of the first roll and the second roll.

- (b) Determine the fraction of possible outcomes in which the sum of the two rolls is at least eight.

Solution: We enumerate all outcomes in which the sum of the two rolls is at least eight:

First Roll	Second Roll
2	6
3	5,6
4	4,5,6
5	3,4,5,6
6	2,3,4,5,6

Counting reveals that the sum of the two rolls is at least eight in $15/36 = 5/12$ of all possible outcomes.

- (c) Determine the fraction of possible outcomes in which the first roll is odd *or* the sum of the two rolls is at least eight.

Solution: We could enumerate all outcomes that meet one of these two criteria. Here we show an alternative solution that uses the set identity $|\mathcal{A} \cup \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}| - |\mathcal{A} \cap \mathcal{B}|$.

Let $\mathcal{A}$ be the set of all outcomes in which the first roll is odd. We know from part (a) that $|\mathcal{A}| = 18$. Let $\mathcal{B}$ be the set of outcomes in which the sum of the

two rolls is at least eight: we know from part (b) that $|\mathcal{B}| = 15$. $\mathcal{A} \cap \mathcal{B}$ is the set of all outcomes in which the first roll is odd *and* the sum of the two rolls is at least eight. In the table in part (b), $\mathcal{A} \cap \mathcal{B}$ contains 6 outcomes with ‘3’ or ‘5’ as the first roll:

First Roll	Second Roll
2	6
3	5,6
4	4,5,6
5	3,4,5,6
6	2,3,4,5,6

Therefore, $|\mathcal{A} \cup \mathcal{B}| = 18 + 15 - 6 = 27$. The fraction of possible outcomes in which the first roll is odd *or* the sum of the two rolls is at least eight is $27/36 = 3/4$.

- (d) Of the outcomes in which the sum of the two rolls is at least eight, what fraction has at least one die showing ‘5’?

Solution: Again, we use the table from part (b). There are 15 possible outcomes in which the sum of the two rolls is at least eight. Out of these, there are 7 outcomes in which at least one roll results in ‘5’, shown in bold below:

First Roll	Second Roll
2	6
3	5,6
4	4, 5 ,6
5	3,4,5,6
6	2,3,4, 5 ,6

Therefore, out of all the possible outcomes in which the sum of the two rolls is at least eight, the fraction in which at least one die shows ‘5’ is $\frac{7}{15}$.

Problem 1.2

Let $f(x, y) = 2x^2 + \sqrt{xy}$ be a function over the domain $(x, y) \in \{1, \dots, n\} \times \{1, \dots, n\}$.

- (a) Suppose we want to evaluate $f(x, y)$ for all possible pairs (x, y) in its domain. Provide an efficient algorithm to perform these computations.

Solution: A simple solution is to iterate over all possible n^2 pairs (x, y) one at a time and compute the value of the function for each pair. Each iteration would require 5 operations (2 multiplications, 1 square, 1 root, and 1 addition), and therefore the entire procedure would require $5n^2$ operations.

Alternatively, we can make use of the fact that the first term $2x^2$ only depends on x and not on y and save some computations by precomputing this term as shown below.

Algorithm 1: Evaluate $f(x, y)$ for all (x, y) pairs in its domain.

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for  $x = 1, \dots, n$  do
     $z \leftarrow 2x^2$ 
    for  $y = 1, \dots, n$  do
         $f(x, y) \leftarrow z + \sqrt{xy}$ 
    end
end

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- (b) Express the total number of operations (additions, multiplications, etc.) that the algorithm in part (a) performs as a function of n .

Solution: We begin by examining the inner loop that iterates over values of y . Each iteration requires 3 operations: 1 addition, 1 root, and 1 multiplication. The inner loop executes n iterations for each value of x .

The outer loop iterates over values of x , and in each iteration it computes $2x^2$ (1 multiply and 1 square) and runs the inner loop once ($3n$ operations). The total number of operations for one iteration of the outer loop is $3n + 2$. The outer loop is repeated n times. The total number of operations is therefore $n(3n + 2) = 3n^2 + 2n$.

As $n \rightarrow \infty$, this algorithm becomes increasingly more efficient relative to the baseline solution that requires $5n^2$ operations.

For each statement below, determine whether it is true or false. If the statement is true, provide a proof. If the statement is false, provide a counterexample.

- (c) The algorithm in part (a) has a time complexity of $\mathcal{O}(n^3)$.

Solution: True. Recall from part (b) that the number of operation we need is $3n^2 + 2n$. Thus, this question is asking if $3n^2 + 2n \in \mathcal{O}(n^3)$. It is true because n^3 grows faster than $n^2 + n$.

We can show this formally by selecting $c = 5$ and examining the behavior of the function $g(n) = 5n^3 - 3n^2 - 2n = 3n^2(n - 1) + 2n(n^2 - 1)$. We observe that $g(n) > 0$ for all $n > 1$, i.e., $3n^2 + 2n < 5n^3$ for all $n > 1$, and therefore $3n^2 + 2n \in \mathcal{O}(n^3)$.

- (d) The algorithm in part (a) has a time complexity of $\mathcal{O}(n^2)$.

Solution: True. Intuitively, this is true because both $3n^2 + 2n$ and n^2 are polynomials of the same order. Indeed, choosing $c = 5$, we can repeat the analysis from part (c) with $g(n) = 5n^2 - 3n^2 - 2n = 2n(n - 1)$, which is strictly positive for $n > 1$, and therefore $3n^2 + 2n \in \mathcal{O}(n^2)$.

Note that the improvement we introduced over the naive implementation in part (a) does not affect the “big O” order of growth complexity of the solution as it does not reduce the leading power of the polynomial.

- (e) The algorithm in part (a) has a time complexity of $\mathcal{O}(\log n)$.

Solution: False. Intuitively, this is false because $\log$ grows slower than a polynomial $3n^2 + 2n$. To show it formally, we define function

$$h(n) = \frac{\log n}{3n^2 + 2n}$$

and observe that $h(n) \rightarrow 0$ as $n \rightarrow \infty$. This implies that for any $c > 0$ there exists n_0 such that

$$\frac{\log n}{3n^2 + 2n} < \frac{1}{c} \quad \text{for all } n > n_0,$$

or

$$3n^2 + 2n > c \log n \quad \text{for all } n > n_0,$$

and therefore $3n^2 + 2n \notin \mathcal{O}(\log n)$.